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Super Weyl transformations in two dimensions

P S Howe†

Department of Physics, University of Lancaster, Lancaster LA1 4YB, UK

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Abstract. A superspace with two commuting and two anti-commuting co-ordinates is discussed with particular emphasis on its superconformal properties. A complete expansion of the *supervierbein* is given and the local supersymmetry transformations of the component fields derived. Super Weyl transformations are defined and it is shown that (2+2)-dimensional superspace is superconformally flat. The spinning string is re-examined and the problems of previous approaches resolved.

1. Introduction

Superspaces—spaces with both commuting and anti-commuting co-ordinates—have proved to be of great value in the formulation of globally supersymmetric field theories (Fayet and Ferrara 1977). In locally supersymmetric theories, e.g. supergravity (Freedman *et al* 1976; Deser and Zumino 1976a), the superspace technique has been of less practical use due to the proliferation of physically redundant component fields and transformations. Nevertheless, the relationship between the conventional space time formalism and the superspace approach is of considerable interest. In four dimensional supergravity, the equations of motion and an action principle have been given in superspace (Wess and Zumino 1977, 1978; Grimm *et al* 1978) and, more recently, a more detailed investigation has been conducted (Brink *et al* 1978). In this paper we deal with a simpler case, namely a superspace with two co-ordinates of each type. We feel that this is a subject worthy of study in its own right since there are considerable simplifications in ordinary two dimensional geometry and it is therefore of interest to see whether the corresponding superspace mimicks this state of affairs.

Our approach to the problem consists of imposing certain 'kinematic' constraints on the supertorsion (analogous to those adopted in four dimensions) (Wes and Zumino 1977, 1978; Grimm *et al* 1978) and then systematically solving for the components of the *supervierbein* by choosing a suitable gauge. We find that the supertorsion and the supercurvature are expressible in terms of one scalar superfield, and that the components of the *supervierbein* and the superconnection may be written down in terms of the *vierbein*, the Rarita–Schwinger field and one auxiliary scalar field. We are then able to derive the locally supersymmetry transformations for these fields. We then turn our attention to the superconformal properties of our space. Having defined super Weyl transformations, we compute their effect on the various geometrical quantities and show that, at least if our kinematic constraints hold, $(2+2)$-dimensional superspace

† Current address: Institute of Theoretical Physics, Göteborg, Fack, S-402 20 Göteborg, Sweden.

is superconformally flat—the analogous result to that obtaining for ordinary two dimensional space. Finally, the spinning string is revisited and discussed in terms of our formalism. An appendix summarising notational conventions is included.

2. General properties

We consider a superspace with co-ordinates $z^M = (x^m, \theta^\mu)$ where m and μ can both take on two values. The x's are the normal commuting co-ordinates whilst the θ's anticommute. Using the standard convention that $(-1)^m$ is plus (minus) one if the index M is bosonic (fermionic), one can express this succinctly by,

$$z^M z^N = (-1)^{mn} z^N z^M. \tag{2.1}$$

At each point in superspace we introduce a set of basis one-forms E^A,

$$E^A = \mathrm{d}z^M E_M{}^A \tag{2.2}$$

where $E_M{}^A$ is the *supervierbein*. This set of frames is required to be Lorentzian in the sense that the tangent space group acts on E^A according to

$$\delta E^A = E^B L_B{}^A \tag{2.3}$$

where

$$L_B{}^A = L\mathrm{E}_B{}^A$$

$$\mathrm{E}_b{}^a = \epsilon_b{}^a;\ \mathrm{E}_b{}^\alpha = \mathrm{E}_\beta{}^a = 0;\ \mathrm{E}_\beta{}^\alpha = \tfrac{1}{2}(\gamma_5)_\beta{}^\alpha.$$

Given a field V^A, transforming under this group as a vector, say, we may define a covariant exterior derivative by

$$\mathrm{D}V^A = \mathrm{d}V^A + V^B \Omega_B{}^A = \mathrm{d}z^M D_M V^A \tag{2.4}$$

where $\Omega_B{}^A$ is the superconnection,

$$\Omega_B{}^A = \mathrm{d}z^M \Omega_{M,B}{}^A. \tag{2.5}$$

In view of the structure of the tangent space group we may take

$$\Omega_B{}^A = \Omega \mathrm{E}_B{}^A. \tag{2.6}$$

Ω transforms inhomogeneously under (2.3);

$$\delta\Omega = -\mathrm{d}L. \tag{2.7}$$

The above definitions may be extended straightforwardly to any p-form transforming in a well-defined way under (2.3). In particular, there are two important two-forms, the torsion and curvature defined by

$$T^A = \mathrm{D}E^A = \tfrac{1}{2}E^C \wedge E^B T_{BC}{}^A, \qquad R_A{}^B = \mathrm{d}\Omega_A{}^B + \Omega_A{}^C \times \Omega_C{}^B = \tfrac{1}{2}E^D \wedge E^C R_{CD,A}{}^B. \tag{2.8}$$

They satisfy the Bianchi identities,

$$\mathrm{D}T^A = E^B \wedge R_B{}^A \tag{2.9}$$

$$\mathrm{D}R_A{}^B = 0. \tag{2.10}$$

Again, the two-dimensional Lorentz group allows us to make the simplification

$$R_A{}^B = F \mathrm{E}_A{}^B; \qquad F = \mathrm{d}\Omega \tag{2.11}$$

in which case (2.10) becomes the obvious identity

$$\mathrm{d}F = 0.$$

In order to reduce the independent components of these tensors, we will employ (2.9) which, written out in full, reads

$$R_{[AB,C]}{}^D = \mathrm{D}_{[A} T_{BC]}{}^D + T_{[AB|}{}^F T_{F|C]}{}^D \tag{2.12}$$

where the square brackets denote generalised anti-symmetrisation, e.g.

$$T_{[AB]} = \tfrac{1}{2}(T_{AB} - (-1)^{ab} T_{BA}).$$

By D_A we mean $E_A{}^M \mathrm{D}_M$ where $E_A{}^M$ is the inverse *supervierbein*

$$E_A{}^M E_M{}^B = \delta_A{}^B.$$

As our kinematic constraints on the supertorsion we take

$$T_{\beta\gamma}{}^a = 2\mathrm{i}(\gamma^a)_{\beta\gamma}, \qquad T_{\beta\gamma}{}^\alpha = T_{bc}{}^a = 0. \tag{2.13}$$

Utilising the Bianchi identity $R_{[\alpha\beta\gamma]}{}^d = 0$ we find

$$T_{\beta c}{}^a = 0. \tag{2.14}$$

Combining the $R_{\alpha\beta,c}{}^d$ and $R_{\alpha\beta,\gamma}{}^\delta$ identities one finds

$$T_{b\gamma}{}^\alpha = \tfrac{1}{4}(\gamma_b)_\gamma{}^\alpha S \tag{2.15}$$

$$F_{\alpha\beta} = -\mathrm{i}(\gamma_5)_{\alpha\beta} S \tag{2.16}$$

where S is a scalar superfield. The identities with $R_{\alpha b,c}{}^d$ and $R_{\alpha\beta,\gamma}{}^\delta$ then yield

$$T_{bc}{}^\alpha = -\tfrac{\mathrm{i}}{4}\epsilon_{bc}(\gamma_5)^{\alpha\beta}\mathrm{D}_\beta S \tag{2.17}$$

$$F_{\alpha\beta} = -\tfrac{1}{2}(\gamma_5\gamma_b)_\alpha{}^\beta \mathrm{D}_\beta S. \tag{2.18}$$

Finally, the $R_{ab,\gamma}{}^\delta$ equation tells us that

$$F_{ab} = \tfrac{\mathrm{i}}{4}\epsilon_{ab}\mathrm{D}_\alpha \mathrm{D}^\alpha S - \tfrac{\mathrm{i}}{4}\epsilon_{ab} S^2. \tag{2.19}$$

Hence we have succeeded in expressing all the components of the curvature and torsion in terms of one scalar superfield S. We remark that if S vanishes, so does the curvature and the space is flat. In this case the *supervierbein* takes the simple form

$$E_m{}^a = \delta_m{}^a;\ E_m{}^\alpha = 0, \qquad E_\mu{}^a = \mathrm{i}\theta^\lambda(\gamma^a)_{\lambda\mu};\ E_\mu{}^\alpha = \delta_\mu{}^\alpha. \tag{2.20}$$

3. The *supervierbein*

Under a general co-ordinate transformation in superspace, $z^M \to z'^M(z)$, the supervierbein changes as follows;

$$E_M{}^A(z) = \frac{\partial z'^N}{\partial z^M} E'_N{}^A(z'). \tag{3.1}$$

Clearly, we can expand z' as a power series in θ^μ;

$$z'^m = x'^m(x) + \theta^\lambda \xi_\lambda{}^m(x) + \tfrac{1}{2}\theta^\lambda\theta_\lambda g^m(x)$$
$$z'^\mu = \xi^\mu(x) + \theta^\lambda \mathrm{D}_\lambda{}^\mu(x) + \tfrac{1}{2}\theta^\lambda\theta_\lambda \eta^\mu(x). \tag{3.2}$$

We observe that under such a transformation $E_\mu{}^A$ transforms in a simple manner and indeed one can use some of the parameters in the expansion (3.2) to translate some of the components of $E_\mu{}^A$ to zero. Explicitly, $\xi_\lambda{}^m$ may be used to gauge away the leading component of $E_\mu{}^a$ and, because of the non-singularity of the *supervierbein* and the transformations, we may use $\mathrm{D}_\lambda{}^\mu$ to set the first term of $E_\mu{}^\alpha$ equal to $\delta_\mu{}^\alpha$. Similarly, g^m and η^μ may be used to transform the antisymmetric parts of the coefficients of θ^λ in the expansions of both $E_\mu{}^a$ and $E_\mu{}^\alpha$ to zero. Thus, (3.1) ensures the existence of a gauge in which, to order θ,

$$E_\mu{}^a = \theta^\lambda E_{\lambda\mu}{}^a; \qquad E_{\lambda\mu}{}^a = E_{\mu\lambda}{}^a$$
$$E_\mu{}^\alpha = \delta_\mu{}^\alpha + \theta^\lambda E_{\lambda\mu}{}^\alpha; \qquad E_{\lambda\mu}{}^\alpha = E_{\mu\lambda}{}^\alpha. \tag{3.3}$$

We may also use the freedom of local Lorentz transformations to simplify Ω_μ. We have

$$\Omega_\mu \to \Omega_\mu - \partial_\mu L \tag{3.4}$$

so that the first term of Ω_μ and the antisymmetric part of the θ coefficient may be gauged away. These choices give us a total of fifteen gauge fixing conditions, reducing the original twenty parameter invariance to five which correspond to x-space reparameterisations, local supersymmetry and local x-space Lorentz invariance. Choosing the first component of $E_m{}^a$ to be the *vierbein*, $e_m{}^a$, and the first component of $E_m{}^\alpha$ to be the $\frac{1}{2}\chi_m{}^\alpha$, the Rarita–Schwinger field, one can then compute the remaining components of the *supervierbein* by a straightforward, albeit somewhat tedious, calculation by using the kinematic conditions (2.13) and their consequences (2.14–18). One finds

$$E_m{}^a = e_m{}^a + \mathrm{i}\bar\theta\gamma^a\chi_m + \tfrac{\mathrm{i}}{4}\bar\theta\theta e_m{}^a A$$
$$E_m{}^\alpha = \tfrac{1}{2}\chi_m{}^\alpha + \tfrac{1}{2}\theta^\mu(\gamma_5)_\mu{}^\alpha\omega_m - \tfrac{1}{4}\theta^\mu(\gamma_m)_\mu{}^\alpha A - \tfrac{3\mathrm{i}}{16}\bar\theta\theta\chi_m{}^\alpha A - \tfrac{1}{4}\bar\theta\theta(\gamma_m)^{\alpha\beta}\psi_\beta \tag{3.5}$$
$$E_\mu{}^a = \mathrm{i}\theta^\lambda(\gamma^a)_{\lambda\mu}, \qquad E_\mu{}^\alpha = \delta_\mu{}^\alpha - \tfrac{\mathrm{i}}{8}\theta\theta\delta_\mu{}^\alpha A$$

where A and ψ are the first and second components of the scalar superfield S. Similarly, one finds for Ω,

$$\Omega_m = \omega_m - \tfrac{\mathrm{i}}{2}\bar\theta\gamma_5\chi_m{}^A + \tfrac{\mathrm{i}}{4}\bar\theta\theta\omega_m{}^A + \tfrac{\mathrm{i}}{4}\bar\theta\theta\epsilon_m{}^n\,\partial_n A + \tfrac{\mathrm{i}}{4}\bar\theta\theta\bar\psi\gamma_5\gamma_m\gamma^n\chi_n + \tfrac{1}{16}\bar\theta\theta\bar\chi\gamma_5\gamma^n\chi_m{}^A \tag{3.6}$$
$$\Omega_\mu = -\tfrac{\mathrm{i}}{2}\theta^\lambda(\gamma_5)_{\lambda\mu}A.$$

The only independent fields in this expansion are the *vierbein*, the Rarita–Schwinger field and the auxiliary scalar field A. From $T_{be}{}^a = 0$ we find

$$\omega_m = -e_{am}\epsilon^{nl}\,\partial_n e_l{}^a + \tfrac{1}{2}\bar\chi_m\gamma_5\gamma^n\chi_n \tag{3.7}$$

whilst from the equation for $T_{bc}{}^\alpha$ (2.17) one obtains

$$\psi = -2\mathrm{i}\epsilon^{mn}\gamma_5\mathrm{D}_m\chi_n - \tfrac{1}{2}\gamma^m\chi_m A. \tag{3.8}$$

Here,

$$\mathrm{D}_m\chi_n = \partial_m\chi_n - \tfrac{1}{2}\omega_m\gamma_5\chi_n$$

is the x-space Lorentz covariant derivative. From (3.7) we observe that the x-space torsion is given by

$$C_{mn}{}^{a} = \mathrm{D}_m e_n{}^{a} - \mathrm{D}_n e_m{}^{a} = \tfrac{1}{2}\bar{\chi}_m \gamma^a \chi_n. \tag{3.9}$$

Finally, we may employ equation (2.19) to obtain the coefficient of $\bar{\theta}\theta$ in the expansion of S. The complete form for S is then

$$\begin{aligned} S &= A + \bar{\theta}\psi + \tfrac{1}{2}\bar{\theta}\theta C \\ C &= R - \tfrac{1}{2}\bar{\chi}_a \gamma^a \psi + \tfrac{\mathrm{i}}{4}\epsilon^{ab}\bar{\chi}_a \gamma_5 \chi_b A - \tfrac{1}{2}A^2 \end{aligned} \tag{3.10}$$

where R *is the* x-space curvature scalar,

$$R = 2\epsilon^{mn}\,\partial_m \omega_n. \tag{3.11}$$

To find the local supersymmetry transformations of the component fields, we must remember that we have used both co-ordinate transformations and tangent space transformation to fix the gauge. Infinitesimally one has

$$\delta E_M{}^{A} = \xi^N\,\partial_N E_M{}^{A} + \partial_m \xi^N E_N{}^{A} + E_M{}^{B} L_B{}^{A}, \qquad \delta\Omega_M = \xi^N\,\partial_N \Omega_M + \partial_M \xi^N \Omega_N - \partial_M{}^{L}. \tag{3.12}$$

From the transformations of $E_\mu{}^{A}$ and Ω_μ we find that the parameters have the expansions

$$\begin{aligned} \xi^m &= f^m - \mathrm{i}\bar{\zeta}\gamma^m\theta + \tfrac{1}{4}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n \\ \xi^\mu &= \zeta^\mu - \tfrac{\mathrm{i}}{2}\bar{\theta}\gamma^m\zeta\chi_m{}^{\mu} - \tfrac{1}{2}\theta^\lambda(\gamma_5)_\lambda{}^{\mu} l - \tfrac{\mathrm{i}}{4}\bar{\theta}\theta(\lambda_5\gamma^m)^{\mu\beta}\zeta_\beta\omega_m - \tfrac{1}{8}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n\chi_m{}^{\mu} \\ L &= l - \tfrac{\mathrm{i}}{2}\bar{\zeta}\gamma_5\theta A - \mathrm{i}\bar{\zeta}\gamma^m\theta\omega + \tfrac{\mathrm{i}}{4}\bar{\theta}\theta\bar{\zeta}\gamma_5\psi + \tfrac{1}{16}\bar{\theta}\theta\bar{\zeta}\gamma^m\gamma_5\chi_m A + \tfrac{1}{4}\bar{\theta}\theta\bar{\zeta}\gamma^n\gamma^m\chi_n\omega_m \end{aligned} \tag{3.13}$$

where the first terms correspond to x-space co-ordinate transformations, local supersymmetry transformations and Lorentz transformations respectively. The supergauge transformations of the basic component fields may now be read off and one finds

$$\delta e_m{}^{a} = \mathrm{i}\bar{\zeta}\gamma^a\chi_m, \qquad \delta\chi_m = 2D_m\zeta - \tfrac{1}{2}\gamma_m\zeta A \qquad \delta A = \bar{\zeta}\psi. \tag{3.14}$$

These are clearly the analogue of the four dimensional supergravity transformations with auxiliary fields (Ferrara and van Nieuwenhuizen 1978) and it is straightforward to check that the algebra closes in a field-independent sense.

4. Super Weyl transformations

We recall that the ordinary Weyl transformations are rescalings of the metric, $g_{mn} \rightarrow f^2 g_{mn}$, and that a space is conformally flat if there exists a co-ordinate system in which the metric tensor is proportional to the flat metric. This latter situation obtains for all two-dimensional spaces which are therefore related by scale transformations to a flat space. In this sections we wish to generalise these ideas to a superspace and to show that any $(2+2)$-dimensional superspace satisfying (2.13) is superconformally flat.

The simplest generalisation of Weyl transformations to superspace would consist of rescalings of the *supervierbein* with appropriate adjustments for the different dimensions involved. However, one can readily verify by direct computation that if one demands that the constraints (2.13) be preserved, then the scaling parameter is restricted to be a constant. A possible way round this is to define super Weyl transformations to have the form

$$\begin{aligned} E_M{}^A \to \hat{E}_M{}^a &= \Lambda(z) E_M{}^a, \\ \hat{E}_M{}^\alpha &= \Lambda^{1/2} E_M{}^\alpha + E_M{}^a \phi_a{}^\alpha(z) \end{aligned} \tag{4.1}$$

where $\phi_a{}^\alpha$ is a spinor parameter to be determined. One finds a consistent scheme is obtained for

$$\phi_a{}^\alpha = -\mathrm{i}(\gamma_a)^{\alpha\beta}\,\mathrm{D}_\beta \Lambda^{1/2} \tag{4.2}$$

and the full list of generalised Weyl transformations is then given by

$$\begin{aligned} \hat{E}_M{}^a &= \Lambda E_M{}^a \\ \hat{E}_M{}^\alpha &= \Lambda^{1/2} E_M{}^\alpha - \tfrac{\mathrm{i}}{2}\Lambda^{-1/2} E_M{}^a (\gamma_a)^{\alpha\beta}\,\mathrm{D}_\beta \Lambda \\ \hat{E}_a{}^M &= \Lambda^{-1} E_a{}^M + \mathrm{i}\Lambda^{-2}(\gamma_a)^{\alpha\beta}\,\mathrm{D}_\beta \Lambda E_\alpha{}^M \\ \hat{E}_\alpha{}^M &= \Lambda^{-1/2} E_\alpha{}^M \end{aligned} \tag{4.3}$$

One can then compute the change in the connection to be

$$\hat{\Omega}_M = \Omega_M + \Lambda^{-1} E_M{}^a \epsilon_a{}^b\,\mathrm{D}_b \Lambda + \Lambda^{-1} E_M{}^\alpha (\gamma_5)_\alpha{}^\beta\,\mathrm{D}_\beta \Lambda \tag{4.4}$$

whilst the superfield S has the transformation

$$\hat{S} = \Lambda^{-1} S + \mathrm{i}\Lambda^{-3}\,\mathrm{D}_\alpha \Lambda \mathrm{D}^\alpha \Lambda - \mathrm{i}\Lambda^{-2} \mathrm{D}_\alpha \mathrm{D}^\alpha \Lambda. \tag{4.5}$$

As in ordinary two-dimensional geometry, we observe that there is no conformally invariant tensor. However, in view of the above scheme, it seems natural to define a super-conformally flat space to be one for which we can choose a co-ordinate system such that the *supervierbein* is related to the flat *supervierbein* by a Weyl transformation i.e.

$$\begin{aligned} E_M{}^a &= \Lambda \bar{E}_M{}^a \\ E_M{}^\alpha &= \Lambda^{1/2} \bar{E}_M{}^\alpha - \mathrm{i}\bar{E}_M{}^a (\gamma_a)^{\alpha\beta} \bar{\mathrm{D}}_\beta \Lambda^{1/2} \end{aligned} \tag{4.6}$$

where $\bar{E}_M{}^A$ is given by (2.20).

We can now show that it is always possible to choose such a gauge for the $(2+2)$-dimensional superspaces under consideration. We decompose the Rarita–Schwinger fields as follows:

$$\chi_m = \gamma_m \phi + \xi_m; \qquad \gamma \cdot \xi = 0. \tag{4.7}$$

Then, because of the two-dimensional identity

$$\gamma_m \gamma^n \gamma^m = 0$$

we may write

$$\xi_m = \gamma^n \gamma_m\,\mathrm{D}_n \alpha$$

for some spinor α. Using a supergauge transformation (3.14) we can therefore set α to zero. In this gauge the x-space torsion $C_{mn}{}^a$ is zero and indeed the χ-contribution to ω_m (3.7) drops out. We may now employ an x-space co-ordinate transformation and an x-space Lorentz transformation to bring $e_m{}^a$ to the conformally flat form

$$e_m{}^a = f \delta_m{}^a. \tag{4.8}$$

Hence one has, from (3.5)

$$E_m{}^a(x, \theta') = f\delta_m{}^a + \mathrm{i}\bar{\theta}'\gamma^a\gamma_m\phi + \tfrac{\mathrm{i}}{4}\bar{\theta}'\theta' fA\delta_m{}^a. \tag{4.9}$$

We may now employ a finite Lorentz transformation with nilpotent parameter $L = -\mathrm{i}\bar{\theta}'\gamma_5\phi$, followed by a co-ordinate transformation $\theta' = f^{1/2}\theta$ to obtain

$$E_m{}^a(x, \theta) = \Lambda\delta_m{}^a \tag{4.10}$$

where

$$\begin{aligned} &\Lambda = f + \mathrm{i}\bar{\theta}\psi + \tfrac{\mathrm{i}}{4}\bar{\theta}\theta b \\ &\psi = f^{3/2}\phi; \qquad b = f^2(A - \mathrm{i}\bar{\phi}\phi) \end{aligned} \tag{4.11}$$

It is a straightforward calculation to verify that repeating this procedure on the remaining components of $E_M{}^A$ yields the desired result (4.6) with Λ given by (4.11). Hence we have succeeded in demonstrating that (2+2)-dimensional superspace is superconformally flat. We remark that this result will not automatically be true in the general case, i.e. if (2.13) is not satisfied. This is because the *supervierbein* and superconnection contain eighty component fields and one only has twenty gauge parameters. However, there are fifty-six constraints in (2.13) and this allows us to describe the geometry by four functions, i.e. a scalar superfield.

5. The spinning string

The spinning string has been solved in x-space by three independent groups (Brink *et al* 1976, Collins and Tucker 1977, Deser and Zumino 1976b) and tackled in superspace in Zumino (1976) and Howe (1977). In the latter paper an *ansatz* was made for the *supervierbein* which produced the correct equations of motion in x-space after the action has been integrated out over the anticommuting co-ordinates. However, the equations in superspace were harder to interpret. On the other hand, in Zumino (1976) the superspace equations of motion were shown to be equivalent to the standard equations of the Neveu–Schwarz–Ramond (Ramond 1971, Neveu and Schwarz 1971) dual model by exploiting the conformal flatness of the superspace. In the latter paper, however, no kinematic conditions were imposed on the supertorsion and hence an additional term in the action was found to be necessary. The problems of the previous two approaches are resolved by the imposition of the kinematic conditions (2.13) which in turn imply that not all of the components of the *supervierbein* may be varied independently.

The spinning string is described in superspace by a scalar superfield V carrying a Lorentz index of the embedding space which we suppress throughout. The natural action is

$$I = \tfrac{1}{2}\int \mathrm{d}^2x\, \mathrm{d}^2\theta \mathscr{L} \tag{5.1}$$

where

$$\mathscr{L} = EL = \tfrac{1}{2}ED_\alpha VD^\alpha V$$

Here, E is the generalised determinant of the *supervierbein*. Variation of I with respect to V yields the equations of motion

$$D_\alpha D^\alpha V = 0 \tag{5.2}$$

whilst variations with respect to the *supervierbein* are expected to yield the constraints of the Neveu–Schwarz–Ramond model. We have

$$\delta\mathscr{L} = [H_a{}^a - 2H_\alpha{}^\alpha]\mathscr{L} - EH_\alpha{}^b D_b V D^\alpha V \tag{5.3}$$

where

$$H_A{}^B = E_A{}^M \delta E_M{}^B. \tag{5.4}$$

Because of (2.13), $H_A{}^B$ satisfies

$$\begin{aligned}\delta T_{BC}{}^A = \mathrm{D}_B H_C{}^A - (-1)^{bc}\mathrm{D}_C H_B{}^A + T_{BC}{}^F H_F{}^A - H_B{}^F T_{FC}{}^A \\ +(-1)^{bc} H_C{}^F T_{FB}{}^A + \psi_{B,C}{}^A - (-1)^{bc}\psi_{C,B}{}^A \\ \psi_{B,C}{}^A = E_B{}^N \delta\Omega_{M,C}{}^A.\end{aligned} \tag{5.5}$$

From $\delta T_{\beta\gamma}{}^a = 0$ one finds

$$H_b{}^a = (\gamma^a\gamma_b)_\alpha{}^\beta H_\beta{}^\alpha - \tfrac{i}{2}(\gamma_b)^{\beta\gamma}\mathrm{D}_\beta H_\gamma{}^a \tag{5.6}$$

and hence

$$H_a{}^a - 2H_\alpha{}^\alpha = -\tfrac{i}{2}(\gamma_b)^{\alpha\beta}\mathrm{D}_\beta H_\alpha{}^\beta. \tag{5.7}$$

The other constraint equations allow us to solve for $\psi_{B,C}{}^A$ and $H_a{}^\beta$ as well as some of the components of $H_\alpha{}^\beta$ in terms of $H_\alpha{}^b$ and the remaining independent parts of $H_\alpha{}^\beta$. However, (5.7) is sufficient for our purposes and one finds, upon employing the integration by parts formula

$$\int \mathrm{d}^2x\, \mathrm{d}^2\theta E \mathrm{D}_A \xi^A (-1)^a = 0 \tag{5.8}$$

valid if $T_{CA}{}^A(-1)^a = 0$, that the constraint equation is

$$(\gamma^a\gamma^b)_\alpha{}^\beta D_\beta V D_a V = 0. \tag{5.9}$$

The left-hand side is just the supercovariant version of the supercurrent which contains amongst its components the energy–momentum and the x-space supercurrent of the system, whose vanishing provides the required dual model constraints. The remaining components contain an auxiliary field which vanishes by virtue of the equations of motion (5.2). Clearly, one can easily reproduce the linearised version of these constraints merely by using the conformal flatness of our space (the action I being super Weyl invariant). In this case the scaling field Λ drops out of the equations and one is left with formally identical equations but with $\mathrm{D}_A V = E_A{}^M \partial_M{}^V$ where $E_A{}^M$ is now the inverse of the flat *supervierbein* (2.20).

Appendix

Co-ordinate indices are given by letters from the middle of the alphabet whilst tangent spaces indices are taken from the beginning. Small latin (greek) letters indicate bosonic (fermionic) components whilst capital letters span both types. The exterior (wedge product) on the basis co-ordinate one-forms is defined by

$$\mathrm{d}z^M \wedge \mathrm{d}z^N = -(-1)^{mn}\,\mathrm{d}z^N \wedge \mathrm{d}z^M \tag{A.1}$$

The exterior derivative d is defined in the standard way, so for a 1-form $W = dz^M W_M$ one has

$$\mathrm{d}W = \mathrm{d}z^M \wedge \mathrm{d}z^N \frac{\partial}{\partial z^N} W_M. \tag{A.2}$$

Exterior differentiation starts from the right so that if W_1 is a p-form and W_2 a q-form

$$\mathrm{d}(W_1 \wedge W_2) = W_1 \wedge \mathrm{d}W_2 + (-1)^q \,\mathrm{d}W_1 \wedge W_2. \tag{A.3}$$

From (A.1) and the definition of the exterior derivative it is simple to verify that

$$\mathrm{d}^2 W = 0 \tag{A.4}$$

for any p-form W.

The bosonic metric is η_{ab}

$$\eta_{ab} = \mathrm{diag}(-1, +1) \quad \text{and} \quad \epsilon_{ab} = -\epsilon_{ba}; \qquad \epsilon_{01} = 1. \tag{A.5}$$

The fermionic 'metric' is $\epsilon_{\alpha\beta'}$

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}; \qquad \epsilon_{12} = 1 = -\epsilon_{21}; \qquad \epsilon_{11} = \epsilon_{22} = 0. \tag{A.6}$$

Spin indices are raised and lowered by $\epsilon_{\alpha\beta}$ according to the rules

$$\chi^\alpha = \epsilon^{\alpha\beta}\chi_\beta; \qquad \chi_a = -\epsilon_{\alpha\beta}\chi^\beta. \tag{A.7}$$

All spinor components are elements of a real Grassmann algebra and the γ matrices are real;

$$\begin{aligned} &(\gamma^0)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \qquad (\gamma^1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &(\gamma_5)_\alpha{}^\beta = (\gamma^0)_\alpha{}^\gamma(\gamma^1)_\gamma{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{A.8}$$

By an expression of the form $\bar{\phi}_1\Gamma\phi_2$ where Γ is any combination of γ matrices and ϕ_1, ϕ_2 are real spinors we mean

$$\bar{\phi}_1\Gamma\phi_2 = \phi_1{}^\alpha\Gamma_\alpha{}^\beta\phi_{2,\beta}. \tag{A.9}$$

Finally, the Fierz rearrangement formula is given by

$$\begin{aligned} &\bar{\phi}_1\phi_2\phi_3 = -\tfrac{1}{2}\sum_i \bar{\phi}_1\Gamma_i\phi_3\Gamma^i\phi_2 \\ &\Gamma_i = 1, \gamma_a, \gamma_5; \qquad \Gamma^i = 1, \gamma^a, \gamma_5. \end{aligned} \tag{A.10}$$

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